Proof that no odd perfect number exists

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Abstract: Using properties of the sum of divisors function, which are derived from its definition, it is proved that every perfect number is even. The general form of perfect numbers is then inferred, and expressed in terms of the sum of divisors function.

Useful formula for the sum of divisors

The divisors of a prime power p^x are the successive powers from 0 to x, so its sum is:

$$\sigma(p^x) = 1 + p + \ldots + p^x \tag{1}$$

It should be easy to see that the one for the next power can be expressed as a function of it in two equivalent recursive ways:

$$\sigma(p^{x+1}) = 1 + p \cdot \sigma(p^x) = \sigma(p^x) + p^{x+1} \tag{2}$$

Grouping terms, the following parametric expression is obtained, which is a usual formula for calculation:

$$\sigma(p^x) = \frac{p^{x+1} - 1}{p - 1} \tag{3}$$

Definition of perfect number

A perfect number N is defined as a natural number whose sum of divisors, $\sigma(N)$, is equal to the double of itself. This can be expressed by the next recursive equation:

$$N = \sigma(N)/2 \tag{4}$$

Extraction of one prime power

Every natural number can be decomposed into the product of powers of prime numbers. A perfect number can not be the power of just one prime, because the only solution then would be $N=2^\infty$, as can be shown from Eq.(3). So we can choose to decompose it into the power of one prime, p^x, and its cofactor, a, which may contain one or more prime powers:

$$N = p^x \cdot a \tag{5}$$

As the sum of divisors function is multiplicative for coprime factors, $\sigma(N)$ can also be decomposed into the corresponding two sums of divisors. Further, at least one of them

has to be even, because of the 2 in the definition of perfect number. We choose p such that $\sigma(p^x)$ is even. As a consequence, $p>2$. Then the original relation, Eq. (4), becomes:

$$p^x \cdot a = \sigma(a) \cdot \frac{\sigma(p^x)}{2} \tag{6}$$

Separation in two equations

Let's now fully factor a into prime powers. Every prime factor q of $\sigma(p^x)/2$ must be a factor of a, and a priori it could be also a factor of $\sigma(a)$. So in the more general way the previous equation can be factored into:

$$p^x \cdot \prod q^{s+r} = \sigma\left(\prod q^{s+r}\right) \cdot \prod q^s \tag{7}$$

Note: a simplified notation is being used on this text, in which the product symbol goes over all the different prime factors q and their corresponding exponents r and s, which are different for each q.

After rearranging components, and using the multiplicative property of the sum of divisors, it becomes:

$$p^x = \prod \frac{\sigma(q^{s+r})}{q^r} = \frac{\prod(1+\ldots+q^{s+r})}{\prod q^r} \tag{8}$$

It should be obvious in the later form that the product in the numerator can not be divided by the product in the denominator unless the denominator is 1, that is, $r=0$ for each r.

As a consequence $\sigma(a)$ has to be coprime to a. Then the Eq.(6) can be decomposed into the following two coupled relations:

$$\begin{cases} \prod q^s = \dfrac{\sigma(p^x)}{2} \\ p^x = \prod \sigma(q^s) \end{cases} \tag{9}$$

Two coprime factors

The first relation contains the sum of divisors of p^x, which is even. The number of divisors of p^x is $(x+1)$, and it has to be also even because every divisor is odd. Therefore the divisors can be grouped two by two, so that a factor $(1+p)$ can be extracted, which is even, so that the first one of the previous equations can be expanded to:

$$\prod q^s = \left(\frac{1+p}{2}\right) \cdot \left(\frac{p^{x+1}-1}{p^2-1}\right) \tag{10}$$

From the general fact that $gcd(A, B) = gcd(A - B, B)$, it is proved without difficulty that

$$\gcd(p^{x+1}-1, p^2-1) = p^2-1 \tag{11}$$

As $(p^2 - 1) = (1 + p)\cdot(p - 1)$, it follows that those two factors in Eq.(10) are coprime. This fact allows to apply Eq.(9) separately for each one of these two cofactors, as it will be done in the next two sections.

Only one remains

It should be noticed that in case of $x=1$, only the first factor on the right of Eq.(10) would remain. Thus, in any case, at least one of the prime powers q^s has to be a factor of $(1+p)/2$. Therefore, p has to be a factor of $\sigma(q^s)$.

Accordingly, the sum of divisors of the second factor on the right of Eq.(10) has to be equal to p^{x-1}, or to a smaller power of p. For the sake of brevity, let's call b to that second factor, and be $y \le x$. Then, a partial application of Eq.(9) to b produces:

$$\begin{cases} b = \dfrac{p^{x+1} - 1}{p^2 - 1} \\ p^{y-1} = \sigma(b) \end{cases} \tag{12}$$

The sum of divisors of any number is always greater than the number itself, unless they both are 1. So the following condition exists:

$$1 \le \frac{\sigma(b)}{b} = \frac{p^{y-1} \cdot (p^2 - 1)}{p^{x+1} - 1} \tag{13}$$

But this inequality can only be fulfilled when $y=x=1$, which implies that $b=1$.

Bound condition

Since $p^x=p$, no more distinct prime factors q are possible, as Eq.(9) then would imply that a prime is a product. That equation is then reduced to:

$$\begin{cases} q^s = \dfrac{1+p}{2} \\ p = \sigma(q^s) \end{cases} \tag{14}$$

From the calculation formula Eq.(3) taken in the limit $s \to \infty$, the following upper bound condition can be established:

$$\frac{q}{q-1} > \frac{\sigma(q^s)}{q^s} = \frac{2 \cdot p}{p+1} \tag{15}$$

Conclusion

The minimum possible value of the term on the right is 3/2, because $p > 2$. Then, as the inequality is not inclusive, it is needed that $q < 3$, so the only possibility is

$$q = 2 \tag{16}$$

Thus 2^s is the only possible factor of $(p + 1)/2$, which is a factor of a, which is a factor of N, so it is concluded that **all perfect numbers are even**.

General form of a perfect number

To complete the case, it can be observed that the two relations of Eq.(14) are really the same one when *q=2*, because the sum of divisors of a power of 2 is:

$$\sigma(2^s) = 2^{s+1} - 1 \tag{17}$$

Then we arrive to the general form of every perfect number *N*:

$$N = 2^s \cdot \sigma(2^s) \mid \sigma(2^s)\, prime \tag{18}$$

which means that **any perfect number must be a power of 2 multiplied by its sum of divisors, with the condition that its sum of divisors must be a prime number**.